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ANALYSIS AND SOLUTION OF THE ILL-POSED INVERSE HEAT CONDUCTION PROBLEM*

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Abstract—The inverse conduction problem arises when experimental measurements are taken in the interior of a body, and it is desired to calculate temperature and heat flux values on the surface. The problem is shown to be ill-posed, as the solution exhibits unstable dependence on the given data functions. A special solution procedure is developed for the one-dimensional case which replaces the heat conduction equation with an approximating hyperbolic equation. If viewed from a new perspective, where the roles of the spatial and time variables are interchanged, then an initial value problem for the damped wave equation is obtained. Since the formulation is well-posed, both analytic and numerical solution procedures are readily available. Sample calculations confirm that this approach produces consistent, reliable results for both linear and nonlinear problems.

NOMENCLATURE

Roman alphabet

C_p,	specific heat;
f, g,	general data functions;
f_1, f_2, g_1, g_2,	specific data functions;
i, j,	discrete mesh indices [cf. equation (20)];
k,	thermal conductivity;
L,	slab length;
M, P,	constants;
n,	summation index;
Q,	heat source density;
r,	radial variable;
t,	time variable;
t_f,	final time;
T,	temperature (dependent variable);
$T_0(x)$,	initial temperature;
$T_1(x)$,	final temperature condition (Section 3);
T_1, T_2,	specific temperature solutions resulting from data f_1, g_1 and f_2, g_2, respectively (Section 2);
u,	temperature variable;
x,	spatial variable (slab geometry);
x', x'',	interior spatial data points.

Greek alphabet

α,	thermal diffusivity;
β,	error term parameter (Section 2); sample problem parameter (Sections 4 and 6);
γ,	parameter [see equation (6a)];
η, ν,	constants;
ξ,	"spatial" variable in equation (7);
ρ,	density;
τ,	"time" variable in equation (7).

Special symbols

$\lvert\cdot\rvert$,	absolute value;
$\lVert\cdot\rVert$,	maximum norm [cf. equation (4)];
Δx, Δt,	numerical stepsizes in time and space, respectively;
x_i, t_j,	discrete mesh points;
$T_{i,j}$,	$T(x_i, t_j)$.

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1. INTRODUCTION

THE CLASSICAL 'direct problem' in heat conduction is to determine the interior temperature distribution of a body from data given on its surface. However, applications arise in which data is not available over the entire surface but is given instead at interior points. In such cases, it is necessary to calculate surface temperatures rather than use them to calculate interior values. This constitutes an 'inverse problem.'

Consider, for example, the illustration in Fig. 1. Boundary conditions given at x' and x'' result in a direct problem for $x' \le x \le x''$ and inverse problems in the regions where $0 \le x \le x'$ or $x'' \le x \le L$. If x' and x'' coincide, then two independent conditions (e.g. temperature and heat flux) must be specified, and two inverse problems exist. If x' and x'' are different points, then the direct problem is solved first and its solution used to obtain boundary conditions for each inverse problem.

Analysis of the direct problem has progressed for almost two centuries, resulting in a wealth of knowledge concerning the behavior of both exact and numerical solution procedures. Because this problem is well-posed, solution computations are straightforward, even for nonlinear problems and irregular geometries. Unfortunately, the same is not true for the inverse problem. The ill-posed nature of this problem not only defies easy solution, but serves to discourage the type of massive study that has accompanied the direct problem.

Present methods of solving the inverse conduction problem include exact, integral, and discrete numerical techniques. Exact and integral methods [1, 2] have

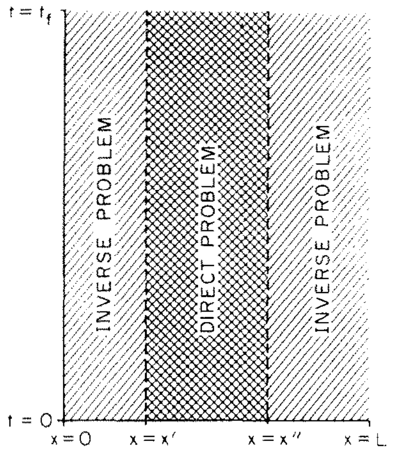

FIG. 1. Diagram of direct and inverse problem regions.

contributed significantly to the theory and understanding of the problem, but are usually restricted to applications with constant physical properties. Discrete approximation by finite differences [3, 4, 5] or finite elements [6] are applicable to nonlinear problems, but the success of these methods is still limited in many cases by the inherent ill-posedness of the problem. Although good results have been obtained in some situations, discrete methods are generally not capable of handling more complicated problems without some corrective action.

As a result, various improvements in conventional discrete methods have been proposed. Garifo, Schrock and Spedicato [4] describe a procedure to approximate the actual solution by solving a sequence of well-posed problems. Good results are obtained for some cases, although the authors note certain situations where the method fails completely. Favorable results have also been achieved by Beck [3] using nonlinear estimation. In this method, accurate surface flux values are determined by a least squares procedure which calculates a correction from previously computed (but possibly inaccurate) temperature values. This is probably the most successful and consistent approach currently in use.

The alternative procedure described in this article consists of developing a complete reformulation of the problem. The inverse problem is closely approximated by a well-posed problem whose solution is easily obtained. In addition, a great wealth of knowledge exists concerning the behavior of this new problem and stable, high order numerical methods are already available.

To better understand the difficulties inherent in the inverse problem, a detailed analysis of its ill-posed nature is carried out in Section 2. The reformulation procedure is then described in Section 3. Section 4 presents four sample problems, representing slab, cylindrical, and spherical geometries, as well as constant and temperature-dependent physical properties. In Section 5, the new solution approach is applied to the sample problems, and the resulting approximations are discretized in order to obtain numerical solutions. These numerical results are reported in Section 6 and compared to the known exact solutions. Finally, Section 7 provides a summary and conclusions.

2. ILL-POSEDNESS OF THE INVERSE HEAT CONDUCTION PROBLEM

The concept of a 'well-posed' problem* was first introduced by J. Hadamard in 1923, and has been developed significantly since then. There is a general consensus that a well-posed problem is one for which a solution 'exists', is 'unique', and depends continuously on given data. This last condition, termed 'stability', ensures that small changes in data will produce small changes in the solution.

The inverse conduction problem has been considered ill-posed by several authors [3, 4]. To undertake a further analysis, consider the heat conduction equation in slab geometry with constant physical properties

$$\frac{\partial T}{\partial t} - \alpha \frac{\partial^2 T}{\partial x^2} = Q(x, t), \quad 0<x<L, \quad 0<t<t_f, \tag{1a}$$

where $\alpha = k/\rho C_p$ is the thermal diffusivity. Referring to Fig. 1, an inverse problem arises when auxiliary conditions occur in the form

$$T(x'', t) = f(t), \tag{1b}$$

$$\frac{\partial T}{\partial x}(x', t) = g(t), \tag{1c}$$

$$T(x, 0) = T_0(x). \tag{1d}$$

For the special case where $x' = x'' = 0$ and $Q(x, t) = 0$, Burggraf [1] has shown that if f and g are infinitely differentiable, then the solution to equations 1(a–c) is given by

$$T(x, t) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(\frac{x^2}{\alpha}\right)^n f^{(n)}(t) - x \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(\frac{x^2}{\alpha}\right)^n g^{(n)}(t). \tag{2}$$

This result is also obtained by Widder [8], who formally proves 'existence' of this solution by showing that both series in equation (2) converge uniformly for bounded t, provided that f and g satisfy

$$|f^{(n)}(t)| \leq M \frac{(2n)!}{L},$$

and

$$|g^{(n)}(t)| \leq P \frac{(2n)!}{L}, \quad n = 0, 1, 2, \ldots$$

for some constants M and P. Uniqueness of the solution (2) follows quickly, since any two solutions must both be equal to the right hand side of equation (2), and therefore to each other.

In many applications, the initial temperature T_0 is

* The information in this paragraph is adapted largely from both the translation editor's preface and the author's preface to the English edition [7].

determined from a steady-state problem using $f(0)$ and $g(0)$ as input. For other situations, T_0 may be determined separately. In any case, if the initial condition T_0 satisfies the compatibility condition

$$T_0(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{x^2}{\alpha}\right)^n f^{(n)}(0) - x\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(\frac{x^2}{\alpha}\right)^n g^{(n)}(0), \quad (3)$$

then the solution at any point depends continuously on the initial temperature. However, even if equation (3) does hold, it will be shown that equation (2) does not depend continuously on f and g.

In examining the stability of the solution (2), it is necessary to measure the differences between two sets of data and the differences between corresponding solutions. For this purpose, consider the maximum-norms for both data and solution,

$$\|f\| = \max_{0\le t\le t_f} |f(t)|,$$

$$\|g\| = \max_{0\le t\le t_f} |g(t)|,$$

$$\|T\| = \max_{\substack{0<x<L\\0\le t\le t_f}} |T(x,t)|. \quad (4)$$

Now, let $T_1(x,t)$ and $T_2(x,t)$ be solutions of equation (2) resulting from the temperature and flux data $f_1(t)$, $g_1(t)$ and $f_2(t)$, $g_2(t)$, respectively. A necessary condition for stability is that the norm $\|T_1 - T_2\|$ can be made arbitrarily small by choosing f_1, f_2, g_1, and g_2 so that the norms $\|f_1 - f_2\|$ and $\|g_1 - g_2\|$ are sufficiently small. This is generally impossible, as can be seen by the following example.

Let $g_1 = g_2 = 0$ and let f_1 be an arbitrary analytic function. For a second data function $f_2(t) = f_1(t) + (1/\beta)\cos\beta^2 t$, $\beta > 0$, the "error term" $(1/\beta)\cos\beta^2 t$ can be made arbitrarily small by choosing β large enough. This forces f_1 and f_2 to be arbitrarily "close," as measured by the norm of equation (4), since

$$\|f_1 - f_2\| = \max_{0\le t\le t_f}\left|\frac{1}{\beta}\cos\beta^2 t\right| = \frac{1}{\beta}.$$

However, as detailed in the Appendix, the corresponding solutions T_1 and T_2 may not be close at all, since

$$\|T_1 - T_2\| \ge \frac{1}{\beta}\left|\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n)!}\beta^{4n}\right| = \frac{1}{\beta}\left|\cosh\frac{\beta}{2}\cos\frac{\beta}{2}\right|. \quad (5)$$

Because the term on the far right is unbounded in β, a small data norm will never result in a small solution norm; hence, the solution (2) is unstable.

3. GENERAL SOLUTION PROCEDURE

A truly satisfactory solution must remain stable even if data is given approximately. Because of the unstable solution dependence on boundary data, the basic inverse formulation (1) does not allow an adequate solution unless special corrective procedures are used.

The alternative procedure presented here completely reformulates the problem so as to produce a well-posed system. This new system will yield a solution which is a good approximation to the desired solution of the ill-posed problem. One aspect of this approach involves a technique called 'quasi-inversion' by Tikhonov and Arsenin [7], who briefly discuss its use for a general nonlinear ill-posed system.

To begin, consider the system

$$\gamma\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} - \alpha\frac{\partial^2 T}{\partial x^2} = 0, \quad 0<x<L,\ 0<t<t_f, \quad (6a)$$

$$T(x,0) = T_0(x), \quad (6b)$$

$$T(0,t) = f(t), \quad (6c)$$

$$\frac{\partial T}{\partial x}(0,t) = g(t), \quad (6d)$$

$$T(x,t_f) = T_1(x), \quad (6e)$$

where γ is a non-negative constant and T_1 is an arbitrary function. If γ is small, then equation (6a) closely resembles equation (1a). In fact, Morse and Feshbach [9] suggest this (the telegraph equation) as a better model of heat conduction, since it does not permit instantaneous transfer of heat as equation (1a) does. For $\gamma > 0$, equation (6a) is hyperbolic and has a solution resembling traveling waves. The characteristics of this equation are lines with slopes $\pm\sqrt{(\gamma/\alpha)}$, as depicted in Fig. 2. Because the domain of dependence for the solution at any point is bounded by the characteristics through that point, the additional boundary condition equation (6e) affects only the shaded region of Fig. 2. For small enough γ, the size of this region becomes insigificant; hence, the effects of equation (6e) can largely be eliminated.† Thus, for small γ, system (6) closely approximates the actual inverse system (1).

In his development of nonlinear estimation, Beck [3] notes that the surface heat flux at time t depends on interior temperature values at times both before and after t. At first glance, this observation may appear somewhat questionable, since it is well known that in direct heat conduction, no temperature may depend on a future boundary value. However, no such rule exists for the inverse problem, where data is given at an interior point instead of at the boundary. In fact, the dependence of surface values on future interior data is a fundamental concept that will allow an efficient, stable solution of system (6).

The hyperbolic system (6) involves only one space variable and is to be solved in the rectangle $0 \le x \le L$,

† If the final time temperature distribution is known, it should certainly be used for $T_1(x)$. In this case, computations in the shaded region of Fig. 2 would produce legitimate temperature values.

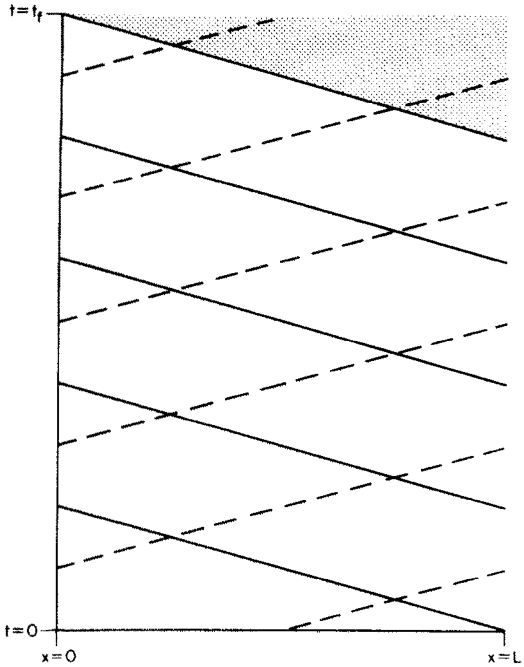

FIG. 2. Characteristic curves of modified system.

$0 \le t \le t_f$. If the roles of the independent variables are reversed, i.e. if x is regarded as the time variable and t as the space variable, then system (6) is identical to the system

$$\alpha \frac{\partial^2 u}{\partial \tau^2} - \gamma \frac{\partial^2 u}{\partial \xi^2} - \frac{\partial u}{\partial \xi} = 0, \quad 0<\xi<t_f, \ 0<\tau<L, \tag{7a}$$

$$u(\xi, 0) = f(\xi), \tag{7b}$$

$$\frac{\partial u}{\partial \tau}(\xi, 0) = g(\xi), \tag{7c}$$

$$u(0, \tau) = T_0(\tau), \tag{7d}$$

$$u(t_f, \tau) = T_1(\tau), \tag{7e}$$

where ξ is considered the space variable and τ the time variable. System (7) is a conventional initial-boundary value problem for a second order hyperbolic equation with constant coefficients. Tychonov and Samarski [10, pp. 107–116] use the Riemann function to derive a closed form solution for this system which is analogous to the D'Alembert solution for the undamped wave equation. The qualities of existence, uniqueness, and continuous dependence on data follow easily from this solution; hence, system (7) represents a well-posed problem.

In summary, the inverse heat conduction problem (1) can be closely approximated by the hyperbolic system (6). If viewed as in system (7), this hyperbolic system is well-posed; hence, the unstable nature and possible data incompatibility of inverse problem (1) are not present in the revised formulation. Another advantage of this approach, to be developed later, is that the numerical solution of the approximating hyperbolic system is efficient and accurate, even for nonlinear problems.

4. SAMPLE PROBLEMS

To further examine this approach, four test problems are considered. All are originally stated in the form of system (1) and have known analytic solutions. Each is then reformulated according to the procedures outlined in Section 3 and solved.

Problem I. Slab geometry, constant properties. This example is used to demonstrate the accuracy that can be obtained for linear problems. For simplicity, the spatial variable is restricted to the interval $0 < x < 1$ and the physical constants k, α are unity. Boundary conditions model an insulated boundary at $x = 0$ and a specified temperature at $x = x''$. Because these conditions are not given at the same point, this problem is well-posed in the region $0 < x < x''$; the formulation in the region $x'' < x < 1$ constitutes an inverse problem. To simplify notation, let $u = u(x, t)$ denote the well-posed solution $(0 < x < x'')$, so that $T(x, t)$ represents only the inverse solution $(x'' < x < 1)$. Both u and T must satisfy the heat conduction equation (1a) and the initial condition (1d) in their respective regions; each must also satisfy the boundary condition (1b). The equation $\partial u/\partial x\,(0, t) = 0$ is the second boundary condition for the well-posed solution u. The second condition [cf. equation (1c)] for T is obtained by applying continuity of heat flux (conservation of energy) at x''; this leads to

$$\frac{\partial u}{\partial x}(x'', t) = \frac{\partial T}{\partial x}(x'', t), \quad 0<t<t_f. \tag{8}$$

Upon solution of the well-posed problem, equation (8) constitutes known boundary data for the ill-posed problem. Selecting the exact solution in both regions to be

$$T(x, t) = \cos x \, e^{\beta t} \tag{9}$$

and substituting into equations (1a), (1b) and (1d), the functions Q, f, and T_0 are determined to be

$$Q(x, t) = (1+\beta)\cos x \, e^{\beta t} \tag{10}$$

$$T_0(x) = \cos x \tag{11}$$

$$f(t) = \cos x'' \, e^{\beta t}. \tag{12}$$

A numerical solution is obtained using equations (10–12) as data in both the well-posed and inverse regions. This solution is then compared with equation (9) for validation.

Problem II. Slab geometry, highly nonlinear. This example, representing heat conduction with a nonlinear source term, is adapted from [4]. Equation (1a) is replaced by

$$\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} = (1+T^2)(1-2T),$$

$$0 < x < 1, \quad 0 < t < t_f, \tag{13}$$

which is solved using the auxiliary conditions 1(b)–(d) with $x'' = 0, f(t) = \tan t, g(t) = 1 + \tan^2 t$, and $T_0(x) = \tan x$. A numerical solution is obtained and then compared with the exact solution

$$T(x, t) = \tan(x + t). \qquad (14)$$

Problem III. Radial geometry, nonlinear, inhomogeneous material. This problem is similar to that encountered in evaluating the heat transfer properties of nuclear reactor fuel rods. A cylinder composed of two materials arranged in concentric regions has the temperature specified at the interface. For simplicity, the radii of the interface and outer boundary are specified as $r = \frac{1}{2}$ and $r = 1$, respectively. Assuming heat conduction only in the radial dimension, the temperature in each region satisfies the equation

$$\rho C_p(T)\frac{\partial T}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left[rk(T)\frac{\partial T}{\partial r}\right] = Q(r, t). \qquad (15)$$

The initial temperature and the internal temperature at $r = \frac{1}{2}$ are also specified, corresponding to conditions (1b) and (1d). Since the problem is independent of azimuthal angle, the boundary condition

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0 \qquad (16)$$

is imposed, which, as in *Problem I*, creates a well-posed problem in the inner region. Upon solution in this region, the continuity of heat flux provides the following interface condition:

$$\left[k(T)\frac{\partial T}{\partial r}\right]_{r=\frac{1}{2}^+} = \left[k(T)\frac{\partial T}{\partial r}\right]_{r=\frac{1}{2}^-}, \qquad (17)$$

which is the second boundary condition for the outer (ill-posed) region. This corresponds to equation (8) in *Problem I*, and can easily be put in the form of equation (1c).

Exact solutions and physical coefficients are chosen to be of the form

$$T(r, t) = \begin{cases} (2 - 5r^2)\,e^{\beta t}, & r \le \frac{1}{2} \\ \cos r\, e^{\beta t}, & r > \frac{1}{2} \end{cases} \qquad (18)$$

$$k(T) = \begin{cases} v(1+T), & r \le \frac{1}{2} \\ 1+T, & r > \frac{1}{2} \end{cases}$$

$$\rho C(T) = \begin{cases} \frac{1}{5}(20-T), & r > \frac{1}{2} \\ \frac{1}{10}(10-T), & r \le \frac{1}{2} \end{cases}$$

Analagous to the development of *Problem I*, these representations are substituted into equations (15), (1d), and (1b) to obtain the heat source and initial temperatures for each region as well as the interface temperature $f(t)$. As described in both previous examples, this "physical data" is used to compute a numerical solution, which is then compared with equation (18) for verification.

Problem IV. Quenching, spherical geometry. If a hot sphere (radius r') at temperature T is suddenly dropped into a liquid which is maintained at the cooler temperature T_1, then the transient temperature distribution in the solid can be determined from the following initial-boundary value problem for direct heat conduction:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial^2(rT)}{\partial r^2}, \quad 0 < r < r', \ t > 0 \qquad (19a)$$

$$T(r, 0) = T_0, \qquad (19b)$$

$$\left.\left(k\frac{\partial T}{\partial r} + hT\right)\right|_{r=r'} = hT, \qquad (19c)$$

$$\frac{\partial T}{\partial r}(0, t) = 0. \qquad (19d)$$

To simplify the analysis, the physical constants are chosen to be $\alpha = h = k = r' = 1$. The solution to system (19) can be expressed as the Fourier sine series

$$T(r, t) = T_1 + \frac{2}{r}(T_0 - T_1) \times \sum_{n=0}^{\infty} \frac{(-1)^n}{\mu_n^2} e^{-\mu_n^2 t} \sin \mu_n r, \qquad (20)$$

$$\mu_n = \frac{(2n+1)}{2}\pi,$$

which is used to obtain the center temperature

$$T(0, t) = T_1 + 2(T_0 - T_1) \sum_{n=0}^{\infty} \frac{(-1)^n}{\mu_n} e^{-\mu_n^2 t}. \qquad (21)$$

Replacing the outer boundary condition (19c) with equation (21) yields an inverse problem, whose solution is determined and compared with the exact solution (20).

5. NUMERICAL APPROXIMATIONS

When data are given at two distinct spatial points, as in *Problems I* and *III*, temperatures in the well-posed region are determined at all time steps before the solution in the ill-posed region is begun. This results in an explicit representation for the interface flux, which is necessary for the solution of the inverse problem. To obtain a discrete approximation in the well-posed region, any numerical method for parabolic equations can be used. The Crank–Nicolson implicit scheme is used for both *Problems I* and *III* in this article, giving excellent results. Details of the method can be found in standard texts on numerical analysis‡; hence, it will not be discussed further in this study.

In order to solve the inverse problems (ill-posed regions of *Problems I* and *III*, entire regions for *Problems II* and *IV*) by the method presented in Section 3, it is necessary to add the term $\gamma(\partial^2 T/\partial t^2)$ to the left-hand sides of equations (1a), (13), (15), and

‡ See, for example, Ames [11] or Richtmyer and Morton [12].

(19a). The resulting differential equations can then be discretized by any conventional method for hyperbolic equations. Consistent with the discussion accompanying system (7), the numerical solution is obtained for all time steps at a given spatial node before any temperature values are computed at the next spatial node. This is in contrast to solution methods for the direct problem which generally solve for all spatial nodes at a given time step before computing any values at the next time step.

As an illustration of the solution procedures used for each inverse problem, consider the modified equation for the ill-posed region of *Problem I*,

$$\gamma \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} = Q(x, t). \tag{22}$$

Introducing the discrete mesh (x_i, t_j) and using the customary notation $T_{i,j}, = T(x_i, t_j)$, equation (22) is discretized using central differences to approximate all derivatives. This results in the following 2nd-order, explicit difference equation:

$$\frac{\gamma}{(\Delta t)^2}(T_{i,j+1} - 2T_{i,j} + T_{i,j-1}) + \frac{1}{2\Delta t}$$

$$\times (T_{i,j+1} - T_{i,j-1}) = \frac{1}{(\Delta x)^2}(T_{i+1,j} - 2T_{i,j}$$

$$+ T_{i-1,j}) + Q(x_i, t_j) + 0[(\Delta x)^2 + (\Delta t)^2]. \tag{23}$$

Equation (23) is tri-level, since temperature values from three different spatial positions appear. It is explicit because only one temperature value at the unknown spatial position x_{i+1} is involved. Courant, Fredricks, and Lewy [13] have shown that this difference equation is convergent (numerically stable and consistent) to the differential equation (22) provided the inequality

$$\gamma \frac{(\Delta x)^2}{(\Delta t)^2} \leq 1 \tag{24}$$

is satisfied. In practice, this condition poses virtually no restriction at all, since γ must be very small from previous considerations.

The discretization of *Problems II, III* and *IV* follows closely the previous discussion for *Problem I*. As in equation (23), the difference equations for these problems, are also 2nd-order, explicit and tri-level. For *Problem II*, solution is restricted to the region where $x + t \leq \pi/2$ since the exact solution (14) becomes infinite if this condition is violated. *Problem IV* is linear, and hence, is stable if equation (24) is satisfied. Were the other problems linear, numerical stability would occur when equation (24) holds for *Problem II* and when

$$\frac{\gamma}{k} \frac{(\Delta x)^2}{(\Delta t)^2} \leq 1 \tag{25}$$

holds for *Problem III* [13]. Although stability conditions for the actual nonlinear problems are not known, equations (24) and (25) were easily satisfied (since γ must be very small) and no computations showed any sign of instability.

6. COMPUTATION RESULTS

Computer programs have been written implementing the numerical procedures of the previous section for each of the sample problems. The exact solutions were also computed and compared with the difference solutions. Results reported here for each problem were calculated using a value of $\gamma = 0.01$, although all runs with $\gamma \leq 0.01$ gave virtually identical results.

The calculated results from all four sample problems showed good accuracy when compared with the exact solutions. As depicted in Table 1, the radial profiles of discrete temperatures for *Problem I* showed agreement with exact values to four decimal places for most points. To illustrate the time behavior of the solutions for *Problem I*, Table 2 compares the surface temperatures and heat fluxes with the exact values. At each time step, the surface heat flux was computed from the temperature distribution using the following 2nd-order, backward difference approximation for the temperature derivative:

$$\left(\frac{\partial T}{\partial x}\right)_{i,j} = \frac{1}{2\Delta x}\left[3T_{i,j} - 4T_{i-1,j} + T_{i-2,j}\right]$$

$$+ O[(\Delta x)^2]. \tag{26}$$

Again, excellent agreement with exact values can be observed.

The results for *Problems II, III* and *IV* are shown in Figs. 3–6. In each case, the relative error between the computed and exact solutions (analogous to columns 4 and 7 of Table 2) is plotted for each time step of a short transient. The calculated fluxes used to produce Fig. 6 were computed using the difference approximation of equation (26). In Fig. 3, the large error near the time point 0.571 is expected, since the exact solution becomes infinite here. Although the computed solution also increases, it cannot match the extremely rapid growth of the exact solution.

The solution to each problem was computed twice, once using exact boundary data and once using data perturbed at each time step by uniformly distributed random errors between $\pm 1\%$. As expected, the calculations using exact data yield results much closer to the exact solutions. However, even the results using the inaccurate data are encouraging, since the random errors are propagated as slight inaccuracies in the surface values, but not as viscious oscillation or unbounded growth.

This is, in fact, exactly what is expected of a good solution method. It is impossible for any method to produce valid surface computations from erroneous data, since any calculation is only as good as the data used. However, it is possible to construct methods that (i) produce calculated values that have the same order of accuracy as 'given' data, and (ii) minimize the effects of instabilities arising in either problem formulation or numerical computation.

Table 1. *Problem I* spatial profile comparison[1] of calculated and exact temperatures solution at selected time values[2]

Time	Radial values 0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
1.5	0.4145	0.4027	0.3898	0.3760	0.3612	0.3456	0.3291	0.3117	0.2936	0.2747	0.2552
	0.4145	0.4027	0.3899	0.3760	0.3613	0.3456	0.3291	0.3118	0.2936	0.2748	0.2552
3.0	0.1958	0.1902	0.1841	0.1776	0.1706	0.1632	0.1554	0.1472	0.1387	0.1298	0.1236
	0.1958	0.1902	0.1842	0.1776	0.1707	0.1633	0.1555	0.1473	0.1387	0.1298	0.1236
4.5	0.0925	0.0899	0.0870	0.0839	0.0806	0.0771	0.0734	0.0696	0.0655	0.0613	0.0569
	0.0925	0.0899	0.0870	0.0839	0.0806	0.0771	0.0734	0.0696	0.0655	0.0613	0.0569

[1] Top line is calculated; bottom line is exact solution.
[2] Uses parameter value $\beta = -0.5$.

Table 2. *Problem I* surface calculations[1]

Time	Calculated temperature	Exact temperature	Relative error[2]	Calculated flux	Exact flux	Relative error[2]
0.0	0.5403	0.5403	0.0000	0.8422	0.8415	−0.0008
0.5	0.4207	0.4208	0.0001	0.6557	0.6553	−0.0005
1.0	0.3277	0.3277	0.0001	0.5106	0.5104	−0.0005
1.5	0.2552	0.2552	0.0001	0.3977	0.3975	−0.0005
2.0	0.1987	0.1988	0.0001	0.3097	0.3096	−0.0005
2.5	0.1548	0.1548	0.0001	0.2412	0.2411	−0.0005
3.0	0.1205	0.1206	0.0001	0.1879	0.1878	−0.0005
3.5	0.0939	0.0939	0.0001	0.1463	0.1462	−0.0005
4.0	0.0731	0.0731	0.0001	0.1139	0.1139	−0.0005
4.5	0.0569	0.0569	0.0002	0.0887	0.0887	−0.0005

[1] Uses parameter values $\beta = -0.5$, $x'' = 0.5$.
[2] Relative error = (exact − calculated)/exact.

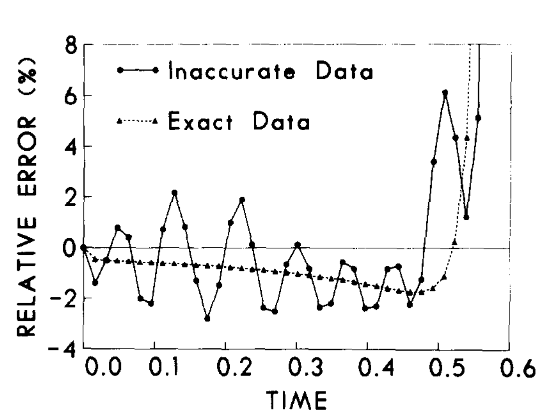

FIG. 3. *Problem II* surface temperature results.

It is evident from Figs. 3–6 that the solution methodology used here satisfies the first condition. The general accuracy of the exact data is preserved in the computation of the surface values. This can be seen by noting that the relative error points for these cases lie generally within one percent of the line where error is zero. Inaccurate data results in surface computations which are also inaccurate by roughly the same order of magnitude. That is, random data errors within 1% produce surface errors generally within 2–3%.

It has been the general thrust of this article to demonstrate that the new solution methodology also meets the second condition. The actual problem solved [system (6)] is well-posed. The difference equation is numerically stable [equations (24) and (25) are satis-

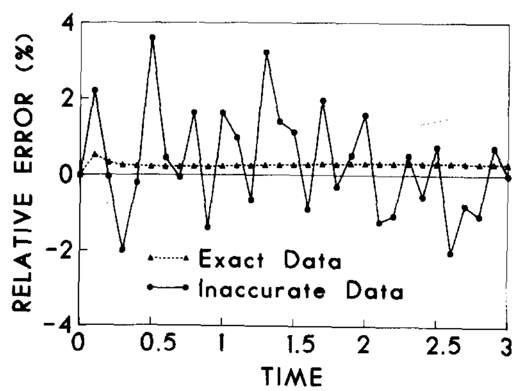

FIG. 4. *Problem III* surface temperature results.

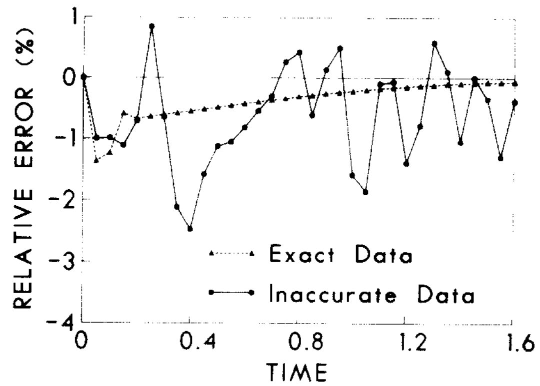

FIG. 5. *Problem IV* surface temperature results.

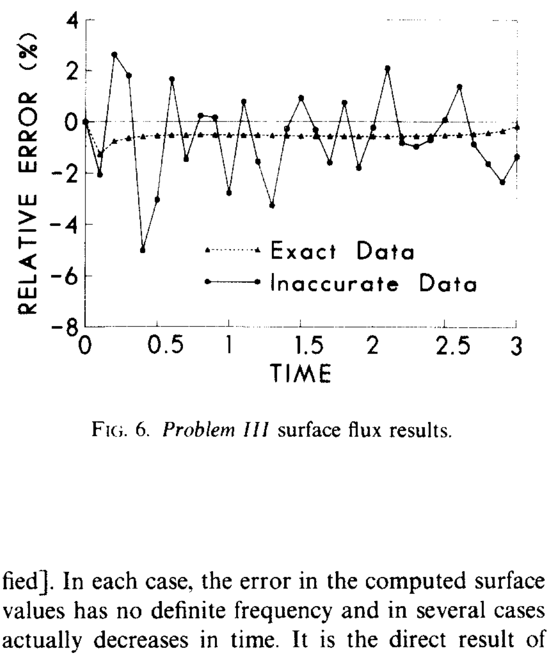

FIG. 6. *Problem III* surface flux results.

fied]. In each case, the error in the computed surface values has no definite frequency and in several cases actually decreases in time. It is the direct result of positive and negative random errors, reflecting only the *inaccuracy* of the data; oscillatory or growth instability is not present. Thus, it is evident that the second condition is indeed satisfied by the solution approach presented here.

The data in most real-world problems contain varying amounts of random errors or noise, which can often be largely eliminated by smoothing techniques. Although this subject can be considered independently, it is worthy of mention here. While smoothing generally results in more accurate data, some perturbations that appear to be inaccuracies might actually be valid components of the data. In any case, the smoothing techniques must maintain or improve the validity of the data itself. The inverse problem solution methodology must then produce surface values that are accurate with respect to whatever data is given. Although several popular approaches have combined smoothing with the inverse solution technique, this, in general, need not be the case.

7. SUMMARY AND CONCLUSIONS

The analysis of the linear inverse formulation in Section 2 showed that the exact solution of system (1) is characterized by discontinuous dependence on data (instability). Nevertheless, most previous solution procedures have attempted to solve system (1) by a direct approach, and certain of these methods have obtained reasonable solution results for some cases. However, the ill-posed nature of the problem implies that direct solution procedures are not capable of consistently producing reliable results.

In order to be assured of a meaningful solution, it was useful to completely reformulate the problem. Thus, the heat conduction operator was approximated by a hyperbolic equation, and was solved numerically for all time steps at each spatial node before progressing to the next spatial node. Although this violates a fundamental principle of 'direct' heat conduction (namely, temperatures at a given time do not depend on temperature values at any future time), it was seen to be a significant factor in devising a solution methodology for the inverse problem. Of particular importance in this approach was the introduction of the coefficient γ [cf. equation (6a)], which was required to be small in order to (i) assure that the governing equation accurately described heat conduction, (ii) reduce the domain of influence of the added final time condition, and (iii) maintain numerical stability. The formulation presented in Section 3 utilized all of these elements to produce a well-posed problem that closely approximated the original inverse problem.

The new solution method was applied to several test problems and the resulting solutions were in good agreement with the known exact solutions. The procedure was applicable to nonlinear problems and to problems involving several material regions. The numerical solution was obtained efficiently and quickly by an explicit, 2nd-order algorithm; solution stability was maintained with insignificant restrictions on stepsizes, even for randomly perturbed data. Thus, the method appears to be applicable to virtually any formulation of this inverse heat conduction problem.

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APPENDIX

Taking the difference of the functions T_1 and T_2 in equation (5) results in

$$T_1 - T_2 = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{x^2}{\alpha}\right)^n \frac{d^n}{dt^n}\left(\frac{1}{\beta}\cos\beta^2 t\right)$$

$$= \frac{1}{\beta}\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n)!}\left(\frac{x^2}{\alpha}\right)^{2n}\beta^{4n}\cos\beta^2 t$$

$$+ \frac{1}{\beta}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(4n+2)!}\left(\frac{x^2}{\alpha}\right)^{2n+1}\beta^{4n+2}\sin\beta^2 t.$$

For $t = 0$, $x^2 = \alpha$, the above reduces to

$$T_1 - T_2 = \frac{1}{\beta}\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n)!}\beta^{4n},$$

and therefore

$$\|T_1 - T_2\| = \max_{\substack{0\le t\le t\\ 0\le x\le L}}\left|T_1(x,t) - T_2(x,t)\right| \geq \frac{1}{\beta}\left|\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n)!}\beta^{4n}\right|.$$

A closed form for the series on the right can be obtained as follows:

$$\cosh\mu\cos\mu = \frac{1}{4}(e^{\mu}+e^{-\mu})(e^{i\mu}+e^{-i\mu})$$

$$= \frac{1}{4}[e^{\mu(1+i)} + e^{\mu(1-i)} + e^{\mu(i-1)} + e^{\mu(-1-i)}]$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\frac{\mu^n}{n!}[(1+i)^n + (1-i)^n + (-1)^n(1-i)^n + (-1)^n(1$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{\mu^{2n}}{(2n)!}[(1+i)^{2n} + (1-i)^{2n}]. \qquad (A.1)$$

Substituting the equalities

$$1 + i = \sqrt{(2i)},\ 1 - i = \sqrt{(-2i)}$$

into equation (A.1) gives

$$\cosh\mu\cos\mu = \frac{1}{2}\sum_{n=0}^{\infty}\frac{\mu^{2n}}{(2n)!}[(2i)^n + (-1)^n(2i)^n]$$

$$= \sum_{n=0}^{\infty}\frac{\mu^{4n}}{(4n)!}(2i)^{2n}$$

$$= \sum_{n=0}^{\infty}\frac{(\sqrt{2}\mu)^{4n}}{(4n)!}(-1)^n.$$

Letting $\beta = \sqrt{2}\,\mu$ gives

$$\cosh\frac{\beta}{\sqrt{2}}\cos\frac{\beta}{\sqrt{2}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(4n)!}\beta^{4n},$$

which is the desired result.

ANALYSE ET SOLUTION DU PROBLEME INVERSE ET MAL POSE DE LA CONDUCTION THERMIQUE

Résumé—Le problème inverse de la conduction se pose quand les mesures expériementales sont faites à l'intérieur du corps et qu'il est souhaité connaître la température et le flux thermique à la surface. On montre que le problème est mal posé car la solution révèle une dépendance instable des fonctions des données. Une procédure spéciale est développée pour le cas monodimensionnel et elle remplace l'équation de la conduction thermique par une équation hyperbolique approchante. Il est montré que si les rôles des variables de temps et d'espace sont interchangés on obtient un problème à valeur initiale pour l'équation d'onde amortie. Puisque cette formulation est bien posée, les procédures analytiques et numériques sont efficaces. Des exemples de calcul montrent que cette approche conduit à des résultats cohérents pour des problèmes linéaires ou non-linéaires.

ANALYSE UND LÖSUNG DES SCHLECHT KONDITIONIERTEN INVERSEN WÄRMELEITUNGS-PROBLEMS

Zusammenfassung — Das inverse Problem der Wärmeleitung tritt auf, wenn experimentelle Messungen im Inneren eines Körpers vorliegen und die Werte von Temperatur und Wärmestrom auf der Oberfläche gesucht sind. Es wird gezeigt, daß diese Aufgabe insofern schlecht konditioniert ist, als die Lösung eine instabile Abhängigkeit von den gegebenen Daten-Funktionen zeigt. Es wird ein spezielles Lösungsverfahren für den eindimensionalen Fall entwickelt, bei welchem die Wärmeleitungsgleichung durch eine approximierende hyperbolische Gleichung ersetzt wird. Aus der Sicht einer neuen Perspektive, bei der die Rollen der Raum- und der Zeitvariablen vertauscht sind, wird ein Anfangswertproblem für die gedämpfte Wellengleichung erhalten. Da diese Formulierung des Problems gut konditioniert ist, stehen sowohl analytische als auch numerische Lösungsverfahren dafür zur Verfügung. Beispielrechnungen bestätigen, daß dieser Ansatz konsistente, verläßliche Ergebnisse für lineare und nichtlineare Probleme liefert.

АНАЛИЗ И РЕШЕНИЕ НЕКОРРЕКТНОЙ ОБРАТНОЙ ЗАДАЧИ ТЕПЛОПРОВОДНОСТИ

Аннотация — Обратная задача теплопроводности возникает в том случае, когда по экспериментальным данным, замеренным внутри тела, необходимо рассчитать температуру и плотность теплового потока на поверхности. Показано, что такую задачу трудно сформулировать корректно, так как решение неустойчиво зависит от заданных экспериментально функций. Предложен специальный метод решения для одномерного случая, в котором уравнение теплопроводности заменяется аппроксимирующим его гиперболическим уравнением. Если рассматривать задачу под другим углом зрения, т. е. поменять ролями пространственные и временные переменные, то можно получить задачу для уравнения, описывающего затухающие волны. В связи с тем, что такая формулировка является корректной, можно легко получить аналитическое и численное решения. На примерах показано, что такой метод дает согласованные и надежные результаты при решении как линейных, так и нелинейных задач.